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Hermite–Birkhoff Interpolation and Monotone Approximation by Splines

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1. INTRODUCTION

There has been a great deal of recent interest in the surprisingly difficult problem of Hermite–Birkhoff interpolation (HBI) by polynomials (for a recent review see Sharma [14]). One of the interesting applications of the known sufficient conditions for poisedness of HBI problems has been in the demonstration of uniqueness of best approximation by monotone polynomials (R. A. Lorentz [8]) and more generally by polynomials with restricted derivatives (Roulier and Taylor [11]).

It was first pointed out in a paper by Karlin and Karon [4] that an HBI result for splines is needed to settle similar monotone spline approximation questions. Karlin and Karon began the study of HBI by splines and since then this problem has been considered by Jetter [3] and Melkman [9]. The main contribution of this paper is sufficient conditions guaranteeing poisedness for certain polynomial spline HBI problems which are general enough to be applied to the monotone spline approximation problem.

The zero counting procedure for splines presented in Section 2 is essentially the zero counting of Schumaker [13]. The specific form and notation for the spline Hermite–Birkhoff interpolation considered is given in the next section. Section 4 explains necessary conditions for poisedness and decomposition results. The main theorem which generalizes the Atkinson–Sharma theorem for polynomials [1] is presented in Section 5 and the application to monotone spline approximation is briefly indicated in the last section.

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2. Fundamental Properties of Polynomial Splines

Suppose $a \equiv \xi_0 < \xi_1 < \cdots < \xi_q < \xi_{q+1} \equiv b$ and integers R_ν with $0 < R_\nu \leqslant m$ for $\nu = 1,..., q$ are given. Let $\mathscr{S} = \mathscr{S}_{m-1,p}(\{\xi_\nu\}_1^q ; \{R_\nu\}_1^q)$ denote the space of real polynomial splines of order m with fixed knots $\{\xi_\nu\}_1^q$, each with multiplicity R_ν, respectively, where $p = \sum_{\nu=1}^q R_\nu$. Thus $g \in \mathscr{S}$ is a piecewise polynomial of degree at most $m - 1$ with $g^{(j)}$ continuous in a neighborhood of ξ_ν for $j = 0, 1,..., m - R_\nu - 1$; $\nu = 1,..., q$. We adopt the convention that elements of $\mathscr{S}$ are always right continuous. Further when we differentiate an element of $\mathscr{S}$ we will extend it to be defined everywhere by right continuity.

Throughout this paper a summation over an empty set is zero. The following facts easily follow from the fundamental theorem of algebra for splines [12], [5]:

(1) $\dim \mathscr{S}_{m-1,p}(\{\xi_\nu\}_1^q ; \{R_\nu\}_1^q) = m + p$.

(2) $\dim\{ g\,|_{[\xi_\ell,\xi_s)}: g \in \mathscr{S}\} = m + \sum_{\nu=\ell+1}^{s-1} R_\nu$ for all $0 \leqslant \ell < s \leqslant q + 1$.

(3) $\dim\{ g^{(j)}: g \in \mathscr{S}\} = m - j + \sum_{\nu=1}^q \min[R_\nu , m - j]$ for all $j = 0, 1,..., m - 1$.

(4) $\dim\{ g^{(j)}\,|_{[\xi_\ell,\xi_s)}: g \in \mathscr{S}\} = m - j + p(j\colon \ell, s)$ for all $j = 0, 1,..., m - 1$ and $0 \leqslant \ell < s \leqslant q + 1$ where we define

$$p(j\colon \ell, s) = \sum_{\nu=\ell+1}^{s-1} \min[R_\nu , m - j].$$

We say N is a *zero set* for $g \in \mathscr{S}$ if either N is an interval (possibly degenerate) on which g vanishes but no larger interval containing N has this property or N is a point where g is discontinuous and changes sign. We define the *multiplicity* $z(N)$ of N as a zero set for g as follows:

(a) If $N = \{t\}$, $t \notin \{\xi_\nu\}_1^q$, then g agrees with a polynomial in a neighborhood of t and $z(N)$ is the multiplicity of t as a zero of that polynomial in the usual sense.

(b) If $N = [a, \xi_s]$ or $[a, \xi_s)$, $s \geqslant 1$, let $z(N) = m + \sum_{\nu=1}^{s-1} R_\nu$.

(c) If $N = [\xi_\ell , b]$, $\ell \leqslant q$, let $z(N) = m + \sum_{\nu=\ell+1}^{q} R_\nu$.

(d) If $N = [\xi_\ell , \xi_s]$ or $[\xi_\ell , \xi_s)$, $0 < \ell < s < q + 1$, there exists an $\epsilon > 0$ such that $g(x) \neq 0$ for any $x \in (\xi_\ell - \epsilon, \xi_\ell) \cup (\xi_s , \xi_s + \epsilon)$. We say g changes sign at N if $g(\xi_\ell - \epsilon/2)\, g(\xi_s + \epsilon/2) < 0$ and does not change sign otherwise. Let

$$z(N) = \begin{cases} m + \sum_{\nu=\ell+1}^{s-1} R_\nu + 1, \text{ if } \left(m + \sum_{\nu=\ell+1}^{s-1} R_\nu\right) \text{ is even and } g \\ \text{changes sign at } N, \\ m + \sum_{\nu=\ell+1}^{s-1} R_\nu + 1, \text{ if } \left(m + \sum_{\nu=\ell+1}^{s-1} R_\nu\right) \text{ is odd and } g \\ \text{does not change sing at } N, \\ m + \sum_{\nu=\ell+1}^{s-1} R_\nu, \text{ otherwise.} \end{cases}$$

(e) If $N = \{\xi_\nu\}$, let $\alpha = \max(\beta, \gamma)$ where β and γ are the multiplicities of ξ_ν as a zero of the polynomials which agree with g in a left and right neighborhood of ξ_ν, respectively. Let

$$z(N) = \begin{cases} \alpha + 1, & \text{if } \alpha \text{ is even and } g \text{ changes sign at } \xi_\nu, \\ \alpha + 1, & \text{if } \alpha \text{ is odd and } g \text{ does not change sign at } \xi_\nu, \\ \alpha, & \text{otherwise.} \end{cases}$$

Let $\{N_i\}_1^r$ denote all of the zero sets of $g \in \mathscr{S}$, ordered in such a way that $\sup N_i < \inf N_{i+1}$, $i = 1,..., r - 1$. Then the total zero count for g is $Z(g) = \sum_{i=1}^r z(N_i)$. This zero counting procedure is a slight improvement on the one given in [13]. The following lemmas can be easily established using the arguments of Schumaker [13].

LEMMA 2.1. *Suppose $g \in \mathscr{S}$ is continuous in an open neighborhood containing the interval* $[\sup N_1, \inf N_2]$ *where N_1 and N_2 are consecutive zero sets of g. Then g' has a zero set N' of odd multiplicity with*

$$\sup N_1 < \inf N' \leqslant \sup N' < \inf N_2.$$

LEMMA 2.2. *Assume $g \in \mathscr{S}$ and* $\max_\nu\{R_\nu\} < m - j$ *for some* $j = 0, 1,..., m - 2$, *thus making $g^{(j)}$ continuous. Then* $Z(g^{(j)}) \leqslant Z(g^{(j+1)}) + 1$.

LEMMA 2.3. *For any $g \in \mathscr{S}$, $Z(g) \leqslant m + p$ with equality if and only if g is the zero spline.*

3. HERMITE-BIRKHOFF INTERPOLATION PROBLEMS

Suppose a fixed spline space $\mathscr{S} = \mathscr{S}_{m-1,p}(\{\xi_\nu\}_1^q; \{R_\nu\}_1^q)$ and interpolation points $X = \{a \leqslant x_1 < x_2 < \cdots < x_k \leqslant b\}$ are given. A matrix $E = \{e_{ij}: i = 1,..., k; j = 0, 1,..., m - 1\}$ will be called a *spline incidence matrix* for X and $\mathscr{S}$ provided $e_{ij} = 0, \pm 1$, or 2, for every i, j, and $e_{ij} = -1$ or 2 only if $x_i = \xi_\nu$ for some $\nu = 1,\ldots, q$ and $j \geqslant m - R_\nu$. The *Hermite–Birkhoff interpolation* (HBI) problem defined by $(E, X, \mathscr{S})$ is: given any values $\{\gamma_{ij}\colon e_{ij} = 1 \text{ or } 2\}$ and $\{\gamma_{ij}^{-}\colon e_{ij} = -1 \text{ or } 2\}$, find $g \in \mathscr{S}$ with

$$\begin{aligned} g^{(j)}(x_i) &= \gamma_{ij} && \text{whenever} && e_{ij} = 1 \text{ or } 2, \\ g^{(j)}(x_i-) &= \gamma_{ij}- && \text{whenever} && e_{ij} = -1 \text{ or } 2. \end{aligned}$$

Recall that we assume right continuity of all splines and their derivatives. Thus a 1 indicates interpolation (or interpolation from the right if discontinuous), a -1 indicates interpolation from the left at a discontinuity, and a 2 indicates separate interpolation for the left and right limits at a discontinuity.

It is extremely helpful to indicate the relationship between the interpolation points X and the knots of the spline space $\mathscr{S}$ when displaying a spline incidence matrix. We do this by adding the following auxiliary lines.

(i) If $x_i < \xi_\nu < x_{i+1}$ with $0 < R_\nu \leqslant m$, we draw a solid line between the i-th and $(i+1)$-th rows extending from the $(m - R_\nu)$-th column to the $(m-1)$-th column. If more than one knot lies between x_i and x_{i+1}, then we draw several lines.

(ii) If $x_i = \xi_\nu$ with $0 < R_\nu \leqslant m$, we enclose in a box the entries in the i-th row from the $(m - R_\nu)$-th column to the $(m-1)$-th column.

EXAMPLE 3.1.

$$\left(\begin{array}{cccc}
1 & 0 & 1 & 0 \\ \cline{2-4}
0 & 0 & 0 & \underline{1} \\ \cline{3-4}
0 & 0 & \multicolumn{1}{|c}{-1} & \multicolumn{1}{c|}{0} \\ \cline{3-4}
1 & 0 & 1 & 0 \\ \cline{2-4}
0 & \multicolumn{1}{|c}{0} & 0 & \multicolumn{1}{c|}{2} \\ \cline{2-4}
0 & 1 & 0 & 0
\end{array}\right)$$

This display indicates that $\mathscr{S}$ has four knots with $x_1 < \xi_1 < x_2 < \xi_2 < x_3 = \xi_3 < x_4 < x_5 = \xi_4 < x_6$, $R_1 = 3$, $R_2 = 1$, $R_3 = 2$, and $R_4 = 3$. Note that an entry equal to -1 or 2 can only occur in a box.

We define $\| E \| = \sum_{i,j} | e_{ij} |$ and this indicates the number of interpolation conditions imposed by E. If an HBI problem $(E, X, \mathscr{S})$ has a unique solution for any given data values, we say the problem is *poised.* Obviously this can happen only if $\| E \| = \dim \mathscr{S} = m + p$, and we say that E is full (for $\mathscr{S}$) when this occurs. Equivalently $(E, X, \mathscr{S})$ is poised when the only solution to the homogeneous problem is the zero spline. When $\| E \| \leqslant m + p$, we say $(E, X, \mathscr{S})$ is *quasi-poised* if the dimension of the solution space for the homogeneous problem is exactly $m + p - \| E \|$.

4. Necessary Condition for Poisedness

We now investigate necessary conditions for poisedness and quasi-poisedness which combine the features of the Polya conditions for HBI by polynomials [1], [2], [6], [4], with the interlacing of knots and interpolation points in Hermite interpolation by splines [12], [5]. These conditions, although stated in a different form, agree with or include previous necessary conditions for spline HBI [4], [3], [9].

Given $g \in \mathscr{S}$, let $g_0, g_1, \ldots, g_q$ be the polynomials which agree with g on the respective knot intervals. We define matrices $E(\eta: \ell, s)$ which are essentially submatrices of E, in such a way that they denote exactly the conditions imposed by $(E, X, \mathscr{S})$ directly upon $g_\ell^{(j)}, \ldots, g_{s-1}^{(j)}$ for $j = \eta, \eta + 1, \ldots, m - 1$. Specifically, for some $\eta = 0, 1, \ldots, m - 1$ and $0 \leqslant \ell \leqslant s \leqslant q + 1$, let $E(\eta: \ell, s) = \{e_{ij}^*: i = k_1, \ldots, k_2; j = \eta, \ldots, m - 1\}$ where $k_1 = \inf\{i: \xi_\ell \leqslant x_i\}$, $k_2 = \sup\{i: x_i \leqslant \xi_s\}$, and

$$e_{ij}^* = \begin{cases} 1, \text{ if } i = k_1, x_i = \xi_\ell, \text{ and } e_{ij} = 1 \text{ or } 2, \\ e_{ij}, \text{ if } x_i \in (\xi_\ell, \xi_s), \\ e_{ij}, \text{ if } i = k_2, x_i = \xi_s, \text{ and } j < m - R_s, \\ 1, \text{ if } i = k_2, x_i = \xi_s, \text{ and } e_{ij} = -1 \text{ or } 2, \\ 0, \text{ otherwise.} \end{cases}$$

It is easy to see that for quasi-poisedness there cannot be more conditions in $E(\eta: \ell, s)$ than $\dim\{g^{(\eta)}|_{[\xi_\ell, \xi_s)}: g \in \mathscr{S}\} = m - \eta + p(n: \ell, s)$. Thus we obtain the following necessary conditions which we call the *local Polya conditions* (LPC) for $(E, X, \mathscr{S})$:

$$\| E(\eta: \ell, s)\| \leqslant m - \eta + p(\eta: \ell, s), \qquad \eta = 0, \ldots, m - 1; \quad 0 \leqslant \ell < s \leqslant q + 1.$$

One of the advantages of our method for displaying incidence matrices is that the (LPC) can be checked in the display. Example 3.1 satisfies these necessary conditions.

When equality occurs in some of these (LPC), then the HBI problem can be decomposed into smaller problems, as has been previously noted [4], [3], [9]. The complete details are tedious but not hard and can be found in [10]. We give only a brief summary here.

Lemma 4.1. *If* $\| E(0: \ell, s)\| = m + p(0: \ell, s)$ *for some* $0 < \ell < s \leqslant q + 1$ *or* $0 \leqslant \ell < s < q + 1$, *then* $(E, X, \mathscr{S})$ *can be split vertically into two or three HBI problems, each defined on a spline space of order still m but with fewer knots than* $\mathscr{S}$. *The "central" one of these smaller problems has incidence matrix* $E(0: \ell, s)$.

LEMMA 4.2. *If $x_i = \xi_\nu$ for some i, ν and $e_{ij} = 1$ for all $j = 0, 1,..., m - R_\nu - 1$, or if $R_\nu = m$ for some ν, then $(E, X, \mathscr{S})$ can also be split vertically into two HBI problems considering fewer knots. $E(0{:}\ 0, \nu)$ and $E(0{:}\ \nu, q + 1)$ will be the incidence matrices for these two smaller problems.*

LEMMA 4.3. *If $\| E(\eta{:}\ 0, q + 1)\| = m - \eta + p(\eta{:}\ 0, q + 1)$ for some $\eta = 1, 2,..., m - 1$, then $(E, X, \mathscr{S})$ can be split horizontally into two HBI problems each defined on spline spaces of smaller order. One of these has incidence matrix $E(\eta{:}\ 0, q + 1)$ and is defined for a spline space of order $m - \eta$ which has dimension $m - \eta + p(\eta{:}\ 0, q + 1)$.*

We further note that any of these decompositions preserve the (LPC) and that if the original problem has a full matrix, then so do all of the smaller problems. Quasi-poisedness of $(E, X, \mathscr{S})$ is equivalent to quasi-poisedness of all of the split problems.

5. SUFFICIENT CONDITIONS FOR POISEDNESS

Jetter [3] and Melkman [9] each propose an analog for splines of the theorem by Atkinson and Sharma [1] giving sufficient conditions for a polynomial HBI problem. The main theorem of this section is stronger than these in that given $(E, X, \mathscr{S})$, it places less restrictions on the incidence matrix when showing poisedness. On the other hand these conditions are more complicated because they involve the knot locations and multiplicities. It should be noted that Melkman includes more general boundary conditions in his development.

Let $(E, X, \mathscr{S})$ indicate a given HBI problem. If $x_i \notin \{\xi_\nu\}_1^q$, then we say that we have a *regular sequence* beginning with e_{ij} of order μ when $e_{ij} = e_{i,j+1} = \cdots = e_{i,j+\mu-1} = 1$ with $e_{i,j-1} = 0$ and $e_{i,j+\mu} = 0$ if either is defined. Also if $x_i = \xi_\nu$, then we say that we have a *regular sequence* beginning with e_{ij} of order μ when $e_{ij} = e_{i,j+1} = \cdots = e_{i,j+\mu-1} = 1$ with $j + \mu \leqslant m - R_\nu$, $e_{i,j-1} = 0$ and $e_{i,j+\mu} = 0$ if either is defined. Further a regular sequence e_{ij},..., $e_{i,j+\mu-1}$ is called *strongly regular* if $e_{i,j+\mu}$ is defined, zero, and, in the case where $x_i = \xi_\nu$, $j + \mu < m - R_\nu$. Thus in our display notation, a regular sequence is a string of ones, none occurring in a box, and a strongly regular sequence has the additional property that a zero entry, which also does not appear in a box, follows the string of ones.

In a similar fashion we define a *left* or a *right sequence* to be a string of successive entries indicating interpolation from the left or right, at least one of which occurs in a box. A sequence is *even* if it has even order and *odd* otherwise.

We say that a regular sequence $e_{i,j}, ..., e_{i,j+\mu-1}$ is *supported* provided there exist integers i_1, j_1, i_2, j_2 with $i_1 < i < i_2$,

$$j_1 < \min[j, \{m - R_\nu: x_{i_1} < \xi_\nu < x_i\}],$$

$$j_2 < \min[j, \{m - R_\nu: x_i < \xi_\nu < x_{i_2}\}],$$

$$e_{i_1,j_1} = 1 \text{ or } 2,$$

and

$$e_{i_2,j_2} = \begin{cases} 1, \text{ if } x_{i_2} \notin \{\xi_\nu\}_1^q, \\ 1, \text{ if } x_{i_2} = \xi_\nu \text{ and } j_2 < m - R_\nu, \\ -1 \text{ or } 2, \text{ if } x_{i_2} = \xi_\nu \text{ and } j_2 \geqslant m - R_\nu. \end{cases}$$

The problem $(E, X, \mathscr{S})$ is called weakly conservative (C) if every supported strongly regular sequence is even.

Note that in our display notation, when searching to find entries e_{i_1,j_1} and e_{i_2,j_2} to show supportedness, no auxiliary line indicating a knot may be crossed. Example 3.1 satisfies (C) because it has no supported strongly regular sequences. In view of the decomposition possible using Lemma 4.3, the difference between weakly conservative and what is usually called conservative in the polynomial case is not significant.

THEOREM 5.1. *Suppose $(E, X, \mathscr{S})$ satisfies* (C), (LPC), *and* $\| E \| = m + p$. *Then it is poised.*

Proof. We use induction on m, the order of the spline space. The theorem is trivially true when $m = 1$. The induction hypothesis, then, is that the theorem is true for any spline space of order strictly less than m, where now $m > 1$. Let $(E, X, \mathscr{S})$ satisfy the hypotheses of the theorem where the order of $\mathscr{S}$ is m.

Without loss of generality, we may assume that none of the hypotheses of Lemmas 4.1, 4.2, or 4.3 hold. If any one did, the problem could be split, each smaller problem would satisfy (C), (LPC) and be full, so we could work with each separately. In particular this means that elements of $\mathscr{S}$ are continuous functions and that $\sum_{i=1}^k e_{i,0} > 1$.

Suppose this problem is not poised. Then there exists a non-trivial $g \in \mathscr{S}$ satisfying the homogeneous problem. Let $i_1 < i_2 < \cdots < i_z$, $z > 1$ be the indices where $e_{i,0} = 1$ so that $g(x_{i_1}) = \cdots = g(x_{i_z}) = 0$. Thus g is not constant and g' must be non-trivial. Let $\{N_w\}_1^r$ denote the zero sets of g in increasing order. By Lemma 2.1, g' has odd zeros $\{N_w'\}_1^{r-1}$ with

$$\max N_w < \inf N_w' \leqslant \sup N_w' < \min N_{w+1}.$$

We construct an incidence matrix E' which describes the zero properties of g'. Initially we let $E' = E(1{:}\,0, q+1)$. Since Lemma 4.3 did not apply to E, we have at this stage that $\| E' \| < m - 1 + p = \dim \mathscr{S}_{m-2,p}(\{\xi_\nu\}_1^q\,; \{R_\nu\}_1^q)$. E' thus defined satisfies (C) and (LPC). We shall attempt to add a condition to E' "between" x_{i_α} and $x_{i_{\alpha+1}}$, $\alpha = 1,..., z-1$, without violating (LPC). Briefly, if $z_0 \geqslant 2$ of the $x_{i_1},..., x_{i_z}$ lie in the same zero set $N_w = [\xi_\ell, \xi_s]$, then the fact that Lemma 4.1 does not apply to E implies that $(z_0 - 1)$ ones can be added to the first column of E' by adding new interpolation points from (ξ_ℓ, ξ_s) without violating the (LPC) and (C).

If x_{i_α} and $x_{i_{\alpha+1}}$ belong to different zero sets of g, then between them lies some N_w', an odd zero set for g'. Either we can add a new interpolation point and a one in the beginning column of E' to denote N_w', we can add a one to the end of the supported strongly regular even sequence if it coincides with N_w', or one of the following three things must happen.

(i) $N_w' = \{\xi_\nu\}$, $\xi_\nu \notin \{x_i\}_i^k$, and $R_\nu = m - 1$.

(ii) $N_w' = \{\xi_\nu\}$, $\xi_\nu = x_i$, and $e_{ij} = 1$ for all $j = 1,..., m - R_\nu - 1$.

(iii) $N_w' = [\xi_\ell, \xi_s]$ or $[\xi_\ell, \xi_s)$ with $\| E'(0{:}\, \ell, s)\| = m - 1 + \sum_{\nu=\ell+1}^{s-1} R_\nu$, where $0 < \ell < s < q + 1$.

If we are able to add $(z - 1)$ ones as described above to construct a full E', then E' and the appropriate interpolation points and spline space satisfy the induction hypothesis. Thus the only solution to the homogeneous problem is the zero spline, contradicting the non-triviality of g'. If not, then some situation described by (i), (ii), or (iii) must occur. Each of these, however, implies that E' as constructed can be decomposed according to Lemma 4.1 or 4.2. Let $(E_1, X_1, \mathscr{S}_1),..., (E_\beta, X_\beta, \mathscr{S}_\beta)$ denote the decomposed problems where a restriction of g' is a non-trivial solution to each homogeneous problem (i.e. the "central" split problem defined on $[\xi_\ell, \xi_s]$ when (iii) occurs is left out). Each of these problems lacks at most two conditions from being full and together they are short a total of $\beta - 1$ conditions. Hence there is at least one $(E_i, X_i, \mathscr{S}_i)$ which is full, and they all satisfy (C) and (LPC). The induction hypothesis applies and yields a contradiction to the homogeneous problem $(E_i, X_i, \mathscr{S}_i)$. The theorem is thus established.

COROLLARY 5.2. *Suppose* $(E, X, \mathscr{S})$ *satisfies* (C) *and* (LPC). *Then it is quasi-poised.*

6. Applications to Monotone Spline Approximation

The motivation for the particular sufficient conditions discussed in the previous section was the following application to the problem of best uniform approximation by monotone splines. Let $0 \leqslant k_0 < k_1 < \cdots < k_w \leqslant m-1$ and $\epsilon_v = \pm 1$, $v = 0, 1, \ldots, w$ be given. Then

$$G = \{g \in \mathscr{S}_{m-1,p}(\{\xi_v\}_1^q; \{R_v\}_1^q): \epsilon_v g^{(k_v)}(t) \geqslant 0, \text{ for all } t \in [a, b], v = 0, 1, \ldots, w\}$$

is called a subset of monotone splines. This generalizes the well-known notion of monotone polynomials (see [7], [8]).

Assume $f \in C[a, b] \backslash G$ with $\epsilon_0 f(t) \geqslant 0$ if $k_0 = 0$. The existence of at least one best uniform approximation to f from G follows easily since G is a closed, convex subset of a finite-dimensional subspace. The following theorem concerning uniqueness can be shown using Theorem 5.1.

THEOREM 6.1. *There exists some knot interval* $[\xi_\ell, \xi_s)$, $\ell < s$, *where all best uniform approximations to* f *from* G *are unique.*

We will give only an outline of how the proof differs significantly from the proof of uniqueness for monotone polynomials [8], [11], [6]. First $g^{(k_v)}$ may be discontinuous (although we assume right continuity). When we select places where the constraints are active, we must include in our selection cases where $g^{(k_v)}(\xi_v-) = 0$, since by continuity elements of G must satisfy $\epsilon_v g^{(k_v)}(\xi_v-) \geqslant 0$. Thus we need to have the capability to specify left and right interpolation conditions when the k_v-th derivative may be discontinuous.

For polynomials, if $g^{(k_v)}$ is not identically zero but $g^{(k_v)}(t) = 0$ for some $t \in (a, b)$, then $g^{(k_v)}$ must have an even zero at t. This leads to the construction of a polynomial HBI problem which has only order 2 supported sequences. The derivative of a spline may be zero on an interval without being trivial. Further $g^{(k_v)}(\xi_v)$ may be zero but $g^{(k_v+1)}$ may be discontinuous at ξ_v (although it must change signs since $g^{(k_v)}$ does not at ξ_v). Thus for the monotone spline problem we construct an HBI problem describing some of the zeros of $g^{(k_v)}$, $v = 0, 1, \ldots, w$ where g is a best uniform approximation to f from G and we can add ones to all supported (in our spline since) strongly regular sequences. The conclusion of partial uniqueness follows in a fashion similar to the way unconstrained best uniform spline approximation yields uniqueness on some knot interval.

Necessary (but not sufficient) extended alternation characterizations similar to those of [6] and more general problems involving restricted derivatives can be handled using Theorem 5.1 as well [10].

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